Week 6; (if time) geometry of curves Math 2240, Spring '24

Problem. 1. When is the sum of indicators 1_A and 1_B an indicator function? The product? Difference? In each case, for what sets?

Problem. 2. Let $f : \mathbb{R} \to \mathbb{R}$ monotone and $a < b$. Show that $g(x) = 1_{[a,b]}(x)f(x)$ is integrable (note f is *not* required to be continuous).

Problem. 3. Give an example of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

- f_n is integrable for each n
- $\lim_n f_n(x) = f(x)$ for all $x \in \mathbb{R}^k$
- f is not integrable

Problem. 4. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Riemann integrable, is $f \circ g$ Riemann integrable?

Geometry of Curves

For now, we'll say a *curve* is the image of a smooth function $\alpha : [0,1] \to \mathbb{R}^3$. A curve is *regular* if $\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$. If $\alpha'(t) = 0$ at some point, then we won't have a good way to define a *tangent line.*

Problem. 1. Using the language of manifolds, describe the tangent line of $p = \alpha(t_0)$ for $t_0 \in [0,1]$ using $\alpha'(t_0)$. Discuss why $\alpha'(t_0) = 0$ gives problems.

The *arc length* of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ starting from t_0 is defined to be

$$s(t) = \int_{t_0}^{t} \|\alpha(s)\| ds$$

where $\|\alpha(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ is the length of the vector $\alpha'(t)$.

Problem. 2. A circular disc of radius 1 in the xy plane rolls without slipping along the x axis. The figure traced out by a fixed point on the circumference of the disk is called a *cycloid.*

- Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points ($t \in \mathbb{R}$ where $\alpha'(t) = 0$).
- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

A curve is parametrized by arc-length if $\|\alpha'(t)\| = 1$ for all $t \in I$ (why?). If α is a curve parametrized by arc length, then the number $\|\alpha'(s)\| = k(s)$ is called the *curvature* of α at s.

Problem. 3. Show the curvature of a parametrized straight line is zero everywhere. Conversely, given $\alpha : I \to \mathbb{R}^3$ show that if $k(s) = \|\alpha'(s)\| = 0$ for all s, then α is a parametrization of a straight line.

If $k(s) \neq 0$, then we can write $\alpha''(s) = n(s)k(s)$, where $n(s)$ is a unit vector in the direction of $\alpha''(s)$. Just like we don't want to deal with $\alpha'(s) = 0$, we also don't want to deal with $\alpha''(s) = 0$, since then we don't have a well-defined normal vector to our curve at s; from now on, we consider α to be at least twice continuously differentiable.

Problem. 4. Show that $\alpha''(s)$ is orthogonal to $\alpha'(s)$. (in this way, $n(s)$ is called the *normal vector* of α at s)

We can just as well write $\alpha'(s) = t(s)$ since $\alpha'(s)$ is a unit vector, and we call $t(s)$ the *unit tangent vector* to α at s. From this, we have $t'(s) = k(s)n(s)$. From $t(s)$ and $n(s)$, we can form a new vector $b(s) = t(s) \times n(s)$ called the *binormal* of α at s. $b'(s)$ measures the amount the plane spanned by the vectors $\{t(s), n(s)\}$, is changing from point to point. Aptly, $\|b'(s)\| = \tau(s)$ is called the *torsion* of α at s.

Problem. Show that $b'(s)$ is normal to $t(s)$. Deduce that $b'(s) = \tau(s)n(s)$ for some function τ. What does the sign of $\tau(s)$ represent?

We could try to now compute $n'(s)$, like we've done for $b(s)$ and $t(s)$, but we wouldn't find any new vectors this time:

Problem. Compute $n'(s)$ and express it in terms of τ, b, k, t.

Remark. From the above work, we now have a set of differential equations describing our curve:

$$\begin{aligned} t' &= kn \\ n' &= -kt - \tau b \\ b' &= \tau n \end{aligned}$$

These are called the *Frenet formulas*. The $t - n$ plane is called the *osculating plane*, and the $n - b$ plane is called the *normal plane*. The inverse $R = 1/k$ of the curvature is called the *radius of curvature* at s.

From these differential equations, we can say that if you hand someone $k(s)$ and $\tau(s)$, we can construct a curve through a point $p \in \mathbb{R}^3$ such that $\alpha(s_0) = p$ and the curvature and torsion of α agree with k and τ.

Problem. Lets look at a circle in $\mathbb{R}^3$ centered at the origin contained in the xy plane.

- Find an arc-length parametrization of the circle.
- Calculate $t(s)$ and $n(s)$, and draw the binormal $b(s)$ for some points on the circle
- Calculate $b'(s)$.
- Show that any curve in $\mathbb{R}^3$ which is fully contained in the xy plane has zero torsion.

Now consider a *helix*, parametrized by $\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}),\ s \in \mathbb{R}$, where $c^2 = a^2 + b^2$.

- Show that s represents arc-length starting at $t = 0$, i.e., $s = \int_0^s |\alpha'(t)|dt$.
- Determine the curvature and torsion of α.
- Describe the osculating plane of α and how it changes from point to point.
- Show that the lines of $\mathbb{R}^3$ containing $n(s)$ (right now, this is a vector "rooted" at $\alpha(s)$) and passing through $\alpha(s)$ meet the z-axis under a constant angle of $\pi/2$.
- (extra) Construct the normal bundle $T^{\perp}M$ where $M = \mathrm{im}(\alpha|_{(0,c)})$. For a given point $p \in M$, attach an ε neighborhood of $(p, 0) \in T^{\perp}M$ given by the subspace topology of $\mathbb{R}^{3\times 3}$. Draw a representation of this ε neighborhood in $\mathbb{R}^3$.

- Now do this for every $p \in M$, and look at the union of all such ε neighborhoods. What geometric object pops out?
- Does the picture for the union of ε neighborhoods of TM look different?

Problem. Show that the torsion τ of a curve α is given by

$$\tau(s) = -\frac{(\alpha'(s) \times \alpha''(s)) \cdot \alpha'''(s)}{|k(s)|^2}$$

(so τ somehow is measuring "third order" effects of a curve, and k is measuring "second order" effects).