Week 4; Dual spaces, gradients. . . Math 2240, Spring '24

Now that we've seen a dual space to a vector space, $V^* = \text{Lin}(V, \mathbb{R})$, we can introduce the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$: The *gradient* of a real-valued function f at a point p is defined as the unique vector $\nabla f(p)$ for which

$$\nabla f(p) \cdot v = Df(p)(v) \text{ for all } v \in \mathbb{R}^n$$

All the work is coming from the dot product. Without it, there isn't a basis-free way to relate $Df(p)$ to a unique vector with a consistent action on all other v.

1. Some quick facts about duals

Let V be an n dimensional vector space, and V^* its dual.

- Show that $\dim(V^*) = \dim(V)$

Let $\langle , \rangle$ be an inner product for V and W (notation suppressed).

- Suppose $\{v_i\}_{i=1}^n$ an orthonormal basis for V, and $\{w_i\}_{i=1}^m$ an orthonormal basis for W, and $L : V \to W$ is a linear transformation. Show that if the matrix of L with respect to the basis $\{v_i\}$ and $\{w_i\}$ is M_{ji}, then the matrix of L^*, M^*_{ji}, related to M_{ji} as $M^*_{ji} = M_{ij}$. (i.e., the matrix of the adjoint of a linear transformation is just the transpose of the original matrix).
- ("Riesz Representation") Show that, for a finite-dimensional inner-product space V, $V \cong V^*$ in the sense that for all $\phi \in V^*$, there exists a unique $v_\phi \in V$ such that $\phi(w) = v_\phi \cdot w$. (can you do it without using a basis? This fact is also true for infinite-dimensional inner-product spaces)

2. Gradients

Now that we know that vectors and functionals are interchangeable through the inner product, its easier to see that if $f : \mathbb{R}^n \to \mathbb{R}$, $Df(p)$ and $\nabla f(p)$ are related by the natural isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$.

In the following, suppose that Suppose $M = f^{-1}(0)$, where 0 is a regular value of $f : \mathbb{R}^n \to \mathbb{R}$, ($M$ is an $n-1$ dimensional manifold in $\mathbb{R}^n$).

- If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x^2/a + y^2/b$, sketch the level curves of f for some values of a and b. Calculate $\nabla f(x, y)$ and graph these vectors atop the level curves.

- In a first multi course, you may have heard that "∇f is orthogonal to the level curves of f." Give meaning to this statement in the language of manifolds.

3. The Tangent Bundle is a Manifold

We saw briefly in class that $TM := \{(p, v) : p \in M, v \in T_pM\}$. Here we'll show that this is *also* a manifold if M is a manifold.

- Suppose that $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ and that $M = f^{-1}(0)$ where 0 a regular value. Using f, construct a new map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2k}$ such that $F^{-1}(0) = TM$ (Hint: the first $n - k$ components should probably be related to f. The remaining $n - k$ components need to somehow encode the fact that if $v \in T_pM \subseteq \mathbb{R}^n$, then $v \in \mathrm{Ker}(Df(p))$.

5. A Manifold with Unusual Tangent Spaces

Denote $\Delta_+^{d-1} := \{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1,\, x_i > 0\ \forall i \in \{1, \ldots, d\}\}$.

- Draw Δ_+^1 and Δ_+^2.

If you squint hard enough, this can be a representation of a probability distribution on d objects, where $P(X = i) = x_i$. Each point in Δ_+^{d-1} describes a different distribution.

- Preliminary: Suppose $U \subseteq M$ is an open subset of a submanifold of $\mathbb{R}^n$ (i.e., open in the subspace topology of M). Argue that U is also a submanifold of $\mathbb{R}^n$.
- Argue that Δ_+^{d-1} is a $d - 1$ dimensional manifold in $\mathbb{R}^d$.
- For $p \in \Delta_+^{d-1}$, describe $T_p\Delta_+^{d-1}$. (Hint: Its probably easiest to find a function for which $U \cap \Delta_+^{d-1} = f^{-1}(1)$ and describe $\ker(Df(p))$.

6. Tangent Spaces and some Derived Objects

We saw that T_pM is a k dimensional vector space in $\mathbb{R}^n$ if M is a k dimensional manifold

- What is the dimension of $(T_pM)^*$? What about $(T_pM)^\perp$?
- If $M = f^{-1}(0)$, we saw that $T_pM = \ker Df(p)$. What is the relationship between $(T_pM)^\perp$ and $Df(p)$? What about $\nabla f(p)$?

7. Some Funnier Bundles

Let M a manifold in $\mathbb{R}^n$ as usual.

- Denote $T^*M = \{(p, \phi) \in \mathbb{R}^{2n} \ : \ p \in M, \, \phi \in (T_pM)^*\}$. Is T^*M a manifold? And of what dimension?
- Denote $T^{\perp}M = \{(p, w) \in \mathbb{R}^{2n} \ : \ p \in M, \, w \in (T_pM)^{\perp}\}$. Is $T^{\perp}M$ a manifold? And of what dimension?
- Denote $UM = \{(p, u) \in \mathbb{R}^{2n} \ : \ p \in M, \, u \in \mathbb{R}^n \, \|u\| = 1\}$. Whats the dimension of UM?

8. Vector Fields?

Now that we know TM is a manifold, we can consider functions $X : M \to TM$ to be C^1 if it plays well with differentiable structures on TM and M. Likewise for $\omega : M \to T^*M$. Such a function X is called a *vector field*, and ω a *covector field*.

- Show that ∇f can be considered a vector field on $\mathbb{R}^n$
- Show that Df can be considered a covector field on $\mathbb{R}^n$