Week 3 (2024-02-07) Math 2240, Spring '24

The inverse and implicit function theorems are building blocks for local statements. They roughly say: If you have a map, and at some point its derivative satisfies some linear algebra property, the map itself satisfies that property on a neighborhood. Since linear algebra is easy, this makes local statements easy if our map is well-behaved.

For example: if you have a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$, and at a point $p \in \mathbb{R}^n$ there exists neighborhoods of p and $f(p)$, $U, V \subseteq \mathbb{R}^n$, such that $f|_U : U \to V$ is one-to-one and onto and has a differentiable inverse, then $Df(p)$ is an isomorphism.

What if we go the other way: suppose $Df(p)$ is an isomorphism. Can we find a differentiable inverse to f? Amazingly, the answer is yes, so long as we force $Df(p)$ to be C^1.

Now suppose you have a map $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$, such that $DF(p)$ is onto at some point. Then in linear algebra terms, we have a freedom to choose arbitrary values of the k coordinates which are in the kernel of $DF(p)$. The rest are then determined by $DF(p)$ (think back to linear algebra, if you have $Az = y$, then each element in $ker\ A$ determines a new solution $z' + x$ where $z' \in (ker\ A)^{\perp}$). Also amazingly, this implies we can find a function $g(p_1, \cdots, p_k) = (p_{k+1}, \cdots, p_n)$ such that $F(x, g(x)) = p$ in a neighborhood of x.

1. How do things go wrong ($\mathbb{R} \to S^1$)

Let $f : \mathbb{R} \to S^1$ be given by $f(x) = x \bmod 2\pi$.

- Take $p = 0$. Find a neighborhood (open set containing 0) such that $f|_U : U \to f(U)$ is one-to-one and onto.
- Take $p = 0$ again. Find a neighborhood U such that $f|_U$ is not one-to-one.
- Can you find a U such that $f|_U$ isn't onto, but also isn't one-to-one?

2. A locally but not globally one-to-one function

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x^2 - y^2, 2xy)$$

1. Show that f is one-to-one on the set $A = \{(x, y) : x > 0\}$. (if $f(x, y) = f(a, b)$, then $\|f(x, y)\| = \|f(a, b)\|$).
2. What is $B = f(A)$?
3. If g is the inverse function of f, find $Dg(0, 1)$.

3. Making Manifolds

Let $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be differentiable, and 0 a regular value. Using the inverse ==function theorem, prove that $F^{-1}(0)$ is a manifold.

4. Inverse $\Rightarrow$ Implicit

Assume inverse function theorem to be true. Prove the implicit function theorem. (Hint: You're given $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Look at "graphs of F" where you only choose a subset of the coordinates. Somewhere you need a full-rank matrix, and the only way to make it is with F and the coordinates given).

5. Even nearby points have solutions

Let $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be C^1, and suppose $f(a) = 0$ and $Df(a)$ is onto. Show that if c is close enough to 0 (quantify?), then $f(x) = c$ has a solution.

6. Non-regular doesn't mean non-manifold

Let $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ have a regular value at 0. Show that $G(x) = F(x)^2$ doesn't have 0 as a regular value, yet $G^{-1}(0)$ is a manifold.

7. (HH 2.10.5)

1. Directly (without implicit fn thm) find where $y^2 + y + 3x + 1 = 0$ defines y implicitly as a function of x
2. Check this agrees with implicit fn thm
3. In what neighborhood of $x = -\frac{1}{2}$ are we guaranteed an implicit function $g(x)$ with $g(-\frac{1}{2}) = \frac{\sqrt{3}-1}{2}$?
4. In what neighborhood of $x = -\frac{13}{4}$ are we guaranteed an implicit function $g(x)$ with $g(-\frac{13}{4}) = \frac{5}{2}$?

8. The form of Dg for implicit function g

Suppose $f(x, y) = 0$ and there exists a g such that $g(x) = y$ and $f(x, g(x)) = 0$ in a neighborhood of x. Find Dg in terms of Df.