Week 2; Some Problems on Analytic Stuff (1-31-24) Math 2240, Spring '24

(To the reader; We definitely didn't get to all of these in recitation! This isn't immediately relevant to the material, so don't worry about studying from these problems. Next week we'll come back to earth to talk a bit about Inverse / Implicit Function Theorem on $\mathbb{R}^n$ to supplement the manifold material)

1.

(Weierstrass) Suppose $f_n(x) \leq M_n$ for all $n \geq 1$ and $x \in A$, $M_n \geq 0$. Suppose further that $\sum_n {}_n$ exists. Show $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in x.

2.

A power series $\sum_{n=1}^{\infty} a_n z^n$ (sometimes denoted $(a_n z^n)$) with $z \in \mathbb{C}$ could converge for some values of z, and diverge for others.

- Suppose $(a_n z^n)$ is absolutely convergent for some z_0. Show that it is absolutely convergent for all $|z| < |z_0|$.

3.

The *radius of convergence* for a power series is a number R such that

- for $|z| < R$, $(a_n z^n)$ converges absolutely
- for $|z| > R$, $(a_n z^n)$ diverges

Show that $\frac{1}{R} = \limsup_n |a_n|^{1/n}$.

4.

A more sane definition for the radius of convergence could be

$$R = \sup_{r \geq 0} \{\text{The sequence } (a_n r^n) \text{ is bounded}\}$$

Prove this is equivalent to the above definition.

5.

Give some examples in $\mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n z^n$ ...

- converges absolutely for all $|z| < 1$

- converges absolutely for all $|z| \leq 1$
- diverges absolutely for all $|z| \geq 1$, but converges for $|z| < 1$
- Converges for all $|z| < 1$, and for some $|z| = 1$, but diverges for some other $|z| = 1$. (hard, may need the "summation by parts" result to show your example works)

7.

A function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is said to be *analytic* if it is equal, for all $z \in D$, to an absolutely convergent power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

[A function that is infinitely smooth, but not analytic (?!)]

Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = e^{-1/x}$ if $x \geq 0$, and $f(x) = 0$ if $x < 0$.

1. Define $f_n(x) = \mathbf{1}_{x \geq 0} e^{-1/x}/x^n$. Show that $f_n'(x) = f_{n+2}(x) - n f_{n+1}(x)$. (the "indicator function of a set $A \subseteq \mathbb{R}$, $\mathbf{1}_A(x)$, is defined to be 1 if $x \in A$, and 0 if $x \notin A$)
2. Take for granted that $f_n(x)$ is continuous at 0. Show that f is C^∞ but not analytic.
3. Sketch $f(x)f(1-x)$
4. Show that $f_n(x)$ is continuous at 0. (l'hospitals probably)

8.

(tricky) Suppose two power series a_n, b_n are absolutely summable and have the same values on a disk $D \subseteq \mathbb{C}$. Show that $a_n = b_n$ for all n.