Week 1 (01-25-24) **Math 2240, Fall '23**

1.

- Let $f(x,y) = \frac{x^2y}{x^2+y^2}$ when $(x,y) \neq (0,0)$, 0 when $(x,y) = 0$. Determine for any $\mathbf{v}$ whether f has a directional derivative in the direction of $\mathbf{v}$ at $(0,0)$. Argue that f is not differentiable. However, show that f is continuous at $(0,0)$.
- Give an example of a function that has all directional derivatives at $(0,0)$ but which isn't differentiable **and** isn't continuous.

2.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x,y) = f(x,y,g(x,y))$$

- Find the (Frechet) derivative DF in terms of the partials of f and g.
- If $F(x,y) = 0$ for all x, y, find $D_1 g$ and $D_2 g$ in terms of the partials of f.

3.

Recall that a linear transformation $A : V \to W$ between normed (possibly infinite-dimensional) vector spaces is bounded if $\sup_{\|x\|=1} \|Ax\| < \infty$. First, show that this is equivalent to the condition that there exists a C such that for all x, $\|Ax\| \leq C\|x\|$ Show that if A is continuous then it is bounded ("recall" a normed space can be equipped with a metric $\|x - y\| = d(x,y)$). Does the converse hold?

4.

- Consider the real numbers $\mathbb{R}$ as a $\mathbb{Q}$ vector space. Show that $1 \in V$ is linearly independent of $x \in V$ if and only if x is irrational.
- Are the real numbers $\mathbb{R}$ considered as a $\mathbb{Q}$ vector space finite dimensional?

5.

- Remember what the adjoint of a linear map is: ...
- State the spectral theorem. Why is an inner product necessary for the spectral theorem to work? Your answer should relate the transpose $A^T : W \to V$ to the adjoint $A* : W* \to V*$.
- Give a matrix or linear map for which the spectral theorem fails to apply.

6+

Recall a metric $d : X \times X \to \mathbb{R}$ assigns a distance between points $x, y \in X$ such that

- $d(x, y) \geq 0$, and equality holds if and only if $x = y$
- (symmetry) $d(x, y) = d(y, x)$
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

Now we take X to be a complete metric space. Suppose we had a continuous function $f : X \to X$ such that, for all $x, y \in X$ there exists an $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

1. Prove that there exists a unique fixed point of f, $f(x_0) = x_0$.
2. Use this fact to show that, if $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $0 < |F'(x)| < 1$, there is a unique solution to the equation $F(x) = 0$.
3. Can you generalize 2. to the case $F : \mathbb{R}^n \to \mathbb{R}^n$?