Spring 2024, 2240 Problem Sheet (Second Half)

Many of these problems are not original: They come variously from our course textbook Hubbard & Hubbard, Munkres's "Analysis on Manifolds," do Carmo's "Differential Forms and Applications," Flanders's "Differential Forms with Applications to the Physical Sciences," and Sjamaar's manifolds notes "Manifolds and Differential Forms," and my predecessor TA's for this course.

Many of these problems are calculations, especially once we get to differential forms. This is on purpose: much of the power of forms lies in their ability to calculate quantities with a physical meaning, but in a completely "automatic" way. Much of the rules of forms (antisymmetry, multilinearity) and their interactions with differentiable functions (pullbacks) are defined for the express purpose of convenient calculations relating to volumes. These problems are meant to highlight their power in 1. computing things we already know, in a more "natural" way, and 2. computing things we need, which would have been difficult to even write down without forms (like the infamous "angle form", or the general Stokes' theorem).

Problem 1. Let $a \in \mathbb{R}^n$ and define $\phi_a(v, w) = \det[a\, v\, w]$. Show ϕ_a is a 2 form, and express it as a linear combination of 2 form basis elements.

Problem 2.
1. Show $\omega \wedge \beta = (-1)^{kl}\beta \wedge \omega$ if $\omega \in \Lambda^k$ and $\beta \in \Lambda^l$.
2. If ϕ is any k form on $\mathbb{R}^3$, show $\phi \wedge \phi = 0$. What about $k > 3$?
3. Does the previous part generalize to forms on $\mathbb{R}^n$? That is, given $n > 0$, ϕ a k form on $\mathbb{R}^n$, is $\phi \wedge \phi = 0$?

Problem 3. Fix $x_0 \in \mathbb{R}^n$ and $\{x_1, \cdots, x_k\}$ linearly independent with $M = \text{span}(\{x_i\}$. Define

$$G = [x_1, \cdots, x_k]^T[x_1, \cdots, x_k],$$

and

$$G^* = [x_0, x_1, \cdots, x_k]^T[x_0, x_1, \cdots, x_k].$$

Define $d(x, M) = \inf_{y \in M} d(x, y)$. Show

$$d(x_0, M)^2 = \frac{\det G^*}{\det G}$$

(Hint: look for an x such that $\|Ax - x_0\|^2$ minimized for a suitable A).

Problem 4. In a usual analysis class, you *start* with the definition of measurable, and then build the theory of integration from there. In this class, we constructed the Lebesgue integral as a sort of "completion" of the Riemann integral. Given the Lebesgue integral, we can actually narrow down what the measurable sets *should* be.

If $A \subset \mathbb{R}^n$ is bounded, we say A is *Lebesgue Measurable* if 1_A is L-integrable. We define

$$m(A) = \int 1_A |dx^n|$$

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If $A \subset \mathbb{R}^n$ is unbounded, we say A is *Lebesgue Measurable* if $1_{A \cap B_R(0)}$ is L-integrable for all $R > 0$ (heuristic: "measurable if any bounded subset of it is measurable"). We define

$$m(A) = \sup_{R>0} m(A \cap B_R(0)).$$

Show:

1. A complement of a measurable set is measurable
2. A countable union of measurable sets is measurable
3. If A, B disjoint and measurable, then $m(A \cup B) = m(A) + m(B)$
4. If $\{A_i\}$ is a countable collection of disjoint measurable subsets, then $m(\cup_i A_i) = \sum_i m(A_i)$.
5. If $D \subset A$, and $m(A) = 0$, then D is measurable and $m(D) = 0$.

Problem 5. If ϕ is a 1-form, and ψ is a 3-form, write $\phi \wedge \psi(v_1, v_2, v_3, v_4$ in terms of ϕ and ψ.

Now let ϕ a 2-form and ψ a 2-form. What's $\phi \wedge \psi$?

Problem 6. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, define

$$\phi_F(v_1, \cdots v_{n-1})(\mathbf{x}) = \det[F(\mathbf{x}), v_1, \cdots, v_{n-1}].$$

Write ϕ_F in the basis $dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}$.

Problem 7. Set $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ a 1 form on $\mathbb{R}^n \setminus \{0\}$. Let's restrict it to $U = \{r > 0, 0 < \theta < 2\pi\}$. Define

$$f(r, \theta) = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(so $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$). Compute $f^*\omega$ on U.

Problem 8. Prove that a bilinear $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is alternating if and only if $\phi(v, v) = 0$ for all v.

Problem 9. Show if $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$, then

$$(dx_{i_1} \wedge \cdots \wedge d_{x_{i_k}})(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & \text{if } i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k \\ 0 & \text{else.} \end{cases}$$

Problem 10. Let $\phi = xdx - ydy$, $\psi = zdx \wedge dy + xdy \wedge dz$, $\theta = zdy$. Compute $\phi \wedge \psi, \theta \wedge \phi \wedge \psi$.

Problem 11. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and ω a k form on R^m with $k > m$, show $f^*\omega = 0$

Problem 12. Let $\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$. Compute $\omega \wedge \omega \wedge \cdots \wedge \omega \equiv \omega^{\wedge n}$.

Problem 13. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ differentiable, and $\omega = dy_1 \wedge dy_m$. Show $f^*\omega = \det(df_p)dx_1 \wedge \cdots \wedge dx_n$.

Problem 14. Let $S^3 \subset \mathbb{R}^4$ be the unit sphere. Show

1. $\operatorname{sign}(dx_1 \wedge dx_3 \wedge dx_4)$ is not an orientation
2. $\Omega_x(v_1, v_2, v_3) = \operatorname{sign}\det[x, v_1, v_2, v_3]$ *is* an orientation

Problem 15. Define $S = \{p = (x, y, z) \,|\, x^2 + y^3 + z^4 = 1\}$. Put an orientation on it.

Problem 16. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$. Define $\Phi_F(v, w) = \det(F(x), v, w)$ ("Flux"), $W_F(v) = F(x) \cdot v$ ("Work"), $Mg_x(v_1, v_2, v_3) = g(x)dx_1 \wedge dx_2 \wedge dx_3(v_1, v_2, v_3)$ ("Mass"). Show

$$\Phi_{F\times G} = W_F \wedge W_G$$

and

$$M_{F\times G} = W_F \wedge \Phi_G = \Phi_F \wedge W_G$$

where $\times$ is the usual cross-product.

Problem 17. Given $T = \{z^2 - (2 - \sqrt{x^2+y^2})^2 = 1$ a torus in $\mathbb{R}^3$, define $\omega = e^{z^2} dx \wedge dy$. Compute $\int_T \omega$.

Problem 18. Find all q forms $\omega = \rho(y,z)dx + q(x,z)dy$ such that

$$d\omega = xdy \wedge dz + ydx \wedge dz$$

Problem 19. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be a radial vector field on $\mathbb{R}^n$. Show $d\Phi_R(x)(v) = n(x) \cdot v$ where $n(x)$ is the unit radial vector at x.

Problem 20. If f is a 1 form on $\mathbb{R}^n$, then show $f(y) = v \cdot y$ for some $v \in \mathbb{R}^n$.

If $T : \mathbb{R}^m \to \mathbb{R}^n$, then from the linear algebra section we can represent $T(v) = Bv$ through a matrix B. What is the matrix of $T^* f$?

Problem 21. Let $\{a_i\}$ be a list of rational numbers in $[0,1]$.

1. Show $f(x) = \sum_{k=1}^{\infty} 2^{-k} \frac{1}{\sqrt{|x-a_k|}}$ is Lebesgue integrable
2. Show $f(x)$ converges for almost every x.
3. Find a particular x for which it converges.

Problem 22. (harder) Let $N \subset \mathbb{R}^n$ be a compact orientable k-dimensional manifold with boundary, and give its boundary the induced orientation.

1. Suppose $\partial N = M_1 \cup M_2$, where M_i are disjoint compact $k-1$ dimensional manifolds without boundary (with one or both non-empty). Show that any $k-1$ form ω such that $d\omega = 0$, we have that
$$\int_{M_1} \omega = \int_{M_2} \omega$$
2. If $\partial N = \emptyset$, then show for any k form η such that there exists a $k-1$ form ω with $d\omega = \eta$, we have
$$\int_N \omega = 0$$

Problem 23. Suppose ω is an $n-1$ form on $\mathbb{R}^n \setminus \{0\}$ such that $d\omega = 0$ and $\int_{S^{n-1}} \omega \neq 0$. Show that ω cannot be exact.

Problem 24. Let M be a $k+l+1$ dimensional manifold without boundary, and ω a k-form, η an l-form, defined on $U \supseteq M$ open. Show that

$$\int_M \omega \wedge d\eta = a \int_M d\omega \wedge \eta.$$

Determine the scalar a.